

Localization-Entropy from Holography on Null-Surfaces and the Split Property

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Abstract

Using the conformal equivalence of translational KMS states on chiral theories with dilational KMS states obtained from restricting the vacuum state to an interval (the chiral inversion of the Unruh-effect) it was shown in a previous publications that the diverging volume (length) factor of the thermodynamic limit corresponds to the logarithmic dependence on a decreasing attenuation length associated with the localization-caused vacuum polarization cloud near the causal boundary of the localization region. Far from being a coincidence this is a structural consequence of the fact that both operator algebras, the global thermal and the locally restricted ground state algebra are of the same unique von Neumann type (monad) which is completely different from that met in Born-localized quantum mechanical algebras. Together with the technique of holographic projection this leads to the universal area proportionality.

The main aim in this paper is to describe a derivation which is more in the spirit of recent work on entanglement entropy in condensed matter physics, especially to that of the replica trick as used by Cardy and collaborators. The essential new ingredient is the use of the split property which already has shown its constructive power in securing the existence of models of factorizing theories.

1 Review of known facts

Whereas quantum mechanics (QM) has reached its conceptual maturity a long time ago, relativistic quantum field theory (QFT) is a sophisticated and computational very demanding theory whose conceptual closure remains still a future project. This is evident from the scarcity of mathematically reliable results about interacting models. After almost four decades of stalemate, there has been some modest progress on the central ontological problem of QFT, namely the existence and nonperturbative constructions of interacting models with non-canonical short distance (strictly renormalizable) behavior.

In a classic paper [1] of the 60s entitled "When does a Quantum Field Theory describe Particles?", the issue of the phase space degrees of freedom, which finally led to the split property, entered QFT for the first time. In that paper it was shown that, contrary to naive analogies claiming that QFT is essentially relativistic QM, the phase space structure of QFT is sufficiently different from QM as to merit attention for a better understanding of the field-particle relation¹ [2]. As the title reveals, the authors conjectured that this property is indispensable for the understanding of asymptotic completeness of particle states i.e. of the equality of the Wigner-Fock space with the Hilbert space defined by the quantum fields. This structure of the Hilbert space and its connection to the local properties of the observables via scattering theory were expected to play an important role in establishing the existence of nontrivial models.

The phase space properties of that paper were later significantly sharpened [3] and used in order to prove that a theory in the ground state with a suitable phase space structure also exists in a thermal equilibrium state [4], in other words one can directly pass from the theory in its ground state representation to the thermal setting without running through a different "Thermo-field" quantization procedure. The nontrivial aspect of the argument lies in overcoming the barrier of the inequivalence of the two representations which results in a difference of the von Neumann type of the operator algebras. This is where the *split property* as a consequence of the phase space requirement plays a crucial role.

Recently it was established that the issue of the asymptotic particle completeness and the mathematical existence of QFTs are indeed inexorably linked; within the class of factorizing models they are consequences of verifiable phase space properties. Hence within this setting, the Haag-Swieca conjecture about phase space properties leading to the complete particle interpretation within a mathematically controlled QFTs was vindicated. Since the conceptual and mathematical tools are very different from the better known Lagrangian quantization and functional integral setting, some additional introductory remarks on this topic may be helpful.

Factorizing models are two-dimensional models whose S matrix is purely elastic, a property which is very atypical in QFT. The only known way in which this can be consistent with the multi-particle matrix elements of S factorize in an appropriate way (consistent with the cluster property and macro-causality) in terms of $S^{(2)}$. Elasticity is a property which is known to contradict the structure of interacting QFTs in higher spacetime dimensions [5]. In this factorizing context the requirements of the old (aborted) S -matrix bootstrap approach really work in the sense that it is possible to classify unitary, Poincaré invariant $S^{(2)}$ with the very nontrivial crossing property [6].

But very different from the dreams the protagonists of the bootstrap had in the 60s, these principles are not only incapable to select a unique theory of everything (TOE), but they rather act in the opposite direction of generating more

¹Interacting fields do not create one-particle states; in fact they do not even create states which contain only a finite number of particles.

models than can be "baptized" by local interaction Lagrangians. The infinitely many bootstrap S-matrices can be arranged into families according to symmetry principles and they possess uniquely associated QFTs whose formfactors were constructed within the bootstrap-formfactor program. An important technical tool was added in form of the Zamolodchikov-Faddeev algebra which represents a specific nonlocal modification of on-shell creation/annihilation operators.

The absence of on-shell particle creation (through scattering) is reminiscent of particle number-conservation in QM. But the characteristic property of interactions in QFT is the presence of vacuum polarization clouds in compactly localized states and less the on-shell reaction².

The generators of the Z-F algebra obtained a physically important spacetime when it was recognized that they play a role in the covariant but nonlocal generating fields of the wedge-localized operator algebras [7]. With this spacetime interpretation it became possible to reformulate the bootstrap-formfactor program in terms of local properties based on the concept of PFGs. This notion of vacuum-polarization-free-generators, i.e. of operators which generate vacuum-polarization-free one particle states if acting on the vacuum, turned out to be extremely useful as a local indicator for the absence/presence of interactions. If a subwedge localized PFG exists in a theory, then it possesses a description in terms of a free field.

PFGs always exist as a consequence of scattering theory within the global algebra which lives on full Minkowski spacetime. Modular operator theory assures the existence of wedge-localized PFGs i.e. the wedge region is the best compromise between fields (as carriers of localization) and particles. In general these wedge generators have very bad domain properties which prevent their successive application to the vacuum. To arrive at a manageable situation one demands temperateness of the wedge-localized PFGs [8] and shows that they lead to pure elastic S-matrix of two-dimensional theories. With a mild additional assumption one then arrives at factorizing theories. In this way one gets a very different spacetime characterization of field theoretic integrability in which vacuum polarization clouds play the central role. The advantage of this construction is that one always maintains the connection with the general physical principles which underlie all QFTs and, last not least, for the first time one obtains an existence proof for noncanonical QFTs [9].

In this context it should be recalled that standard methods of showing the existence of pointlike covariant off-shell field generators in terms of on-shell Z-F operators remained inconclusive since formfactors are at best *bilinear forms of would-be operators*; to obtain the existence of bona fide operators and their algebras one has to use alternative methods which avoid the use of pointlike field coordinatization and address the nontriviality of the local algebras in a more intrinsic manner. The experience with divergent perturbative series casts doubts about the convergence status of the two known formally exact nonperturbative series representation of interacting quantum fields namely the Glaser-Lehmann-

²The relativistic QM of *direct interactions* allows the introduction of creation channels "by hand", whereas it is not possible to add vacuum polarization, i.e. there is no passage from the relativistic direct interaction setting to QFT.

Symanzik (GLZ) series in terms of incoming free fields and the closely related series³ for fields in factorizing models in terms Zamolodchikov-Faddeev (Z-F) creation/annihilation operators of representation of Heisenberg fields. Hence the algebraic existence proof is much more than an elegant reformulation of an already existing result. We will return to these ontological issues of interacting QFTs within a more detailed setting in the third section.

The algebraic existence proof for two-dimensional factorizing QFTs uses the simplicity of temperate PFGs as generators of wedge-algebra in an essential way. A perturbative construction of wedge generators in higher dimensions, where no temperate wedge-localized PFGs are available, is an interesting open problem. But it would almost certainly re-introduce those unsolved convergence problems connected with the perturbative series. Its main advantage is expected to be the enlargement the realm of renormalizable interactions (finite-parametric QFTs) through avoidance of pointlike fields. In [17] it was shown that already the use of covariant stringlike localized free fields instead of pointlike fields does improve the short distance behavior so that many more interactions acquire the formal renormalizability status in terms of power counting. One expects that starting with wedge-like localized generators one reaches the frontier of renormalizability i.e. the most general class of interactions for which perturbation theory remains polynomially bounded and permits a finite parametric description which is stable under the action of some suitable defined renormalization group.

A promising rigorous *nonperturbative* idea in higher dimensions consists in working with *holographic mappings of wedge-localized algebras*. Holographic projections in which a localized bulk algebra is projected onto the null horizon of the causally completed bulk lead to significant simplifications in the description of localization-caused properties as energy and entropy densities. Naturally simplifications which do not modify the content of models can only consist in looking at a model from a different vantage point, so that certain aspects one wants to focus on become simpler at the expense of complicating other aspects. QFT still hides many surprises but there are no miracles.

The most prominent alternative approach with this aim is quite old and known under the name of "lightcone quantization" (closely connected to the " $p \rightarrow \infty$ frame" method). The name suggests that it was originally thought of as an alternative quantization to the standard Lagrangian approach from which one expected a simpler understanding of certain high-energy aspects. Since a change of quantization generally leads to a change of the QFT model, this raises the question if, and in what way, the new method is conceptually as well as computationally related to the standard formulation [10][11], a problem which unfortunately was not addressed in most papers.

Lightcone quantization or (in the more appropriate terminology used in this paper) *lightfront restriction* suffers the same formal limitations as equal time commutation relations, namely it becomes meaningless in the Wightman

³The convergence of these series within the Wigner-Fock space of the asymptotic particles would justify the interpretation of formfactors as multi-particle matrix elements of operators in the W-F Hilbert space.

representation using x-space correlation functions (Wightman functions) whenever the wave function renormalization constant Z ceases to be finite and non-vanishing.

The fact that in these cases of infinite wave function renormalization the canonical equal time formalism breaks down has no serious consequences since physical requirements in QFT do not demand that the commutation relations of pointlike fields ought to allow an equal time restriction. Within the perturbative setting this is fully taken into account in the so called *causal perturbation theory* which is only subject to the more general and much weaker requirement of *renormalizability*.

In the following section we use the simplicity of lightfront restriction for free fields in order to explain some important concepts which were absent in the old lightcone quantization work and which have no natural formulation in Lagrangian quantization. In the same section some algebraic concepts, which highlight the completely different nature of localized algebras in QFT from those in QM, will be introduced. The distinction between quantum mechanical entanglement and the thermal aspects of localized algebras in QFT (which being monads have no pure states [15] at all), whose causal horizon is surrounded by a vacuum polarization cloud with an attenuation length ε depending on the variable splitting arrangement, are important manifestations of this significant conceptual difference between QM and QFT.

The third section presents the algebraic holography in the presence of interactions. In that case the connection between the *algebraic holography*, which uses the notions of *modular inclusions* and *modular intersections* of wedge algebras, and the (at the present time less rigorous) *projective holography* of pointlike fields (which requires a mass-shell representation of the interacting fields in terms of an infinite series in the incoming or the Z-F creation/annihilation operators with an unclear convergence status) still needs further clarification.

Some observations, which suggest the way in which the two holographies are related, can be made in the setting of factorizing models and will be presented in the fourth section. There we also comment on the conceptual difference between the holographic projection and the critical limit theory of the bulk (the representative of the universality class) which is known to be a conformal QFT in a different Hilbert space. This has an interesting but still somewhat speculative relation to the issue of Zamolodchikov's deformation of chiral models into factorizing theories.

In the fifth section we remind the reader of the close structural algebraic relation of the thermodynamic limit of a heat bath system and the thermal aspects of localization in QFT. We show that for chiral algebras the length (= one-dimensional volume) factor passes via conformal covariance to the logarithmic attenuation factor. The transversely extended chiral theory, which results from holographic projection of the bulk, turns out to be the *raison d'être for the universal area proportionality of localization-entropy*.

In order to arrive at a completely intrinsic alternative derivation via direct use of the split property, we adapt the replica idea of the derivation of localization-entropy as used by the condensed matter physicists [30] to the

present setting. The result is a conceptually transparent implementation of the replica trick which avoids the very artful but nevertheless metaphoric functional integral representations which will be presented in the sixth section.

The concluding section contains some speculative remarks about the future of QFT which are of a more philosophical nature

2 Holographic projection for free fields, an illustrative example

The shortcomings of the old "lightcone quantization" become more explicit if one compares it with its modern successor which, for reasons which become obvious, will be referred to as "lightfront holography" (LFH). The main idea of LFH can be illustrated with the help of the mass shell representation of the free scalar field $A(x)$ and its restriction to the wedge region [12][28]

$$A(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int (e^{ipx} a^*(p) + h.c.) \frac{d^3 p}{2p_0} \quad (1)$$

$$A_W(r, \chi; x_\perp) \equiv A(x)|_W = \frac{1}{(2\pi)^{\frac{3}{2}}} \int (e^{im_{eff}rch(\chi-\theta)+ip_\perp x_\perp} a^*(\theta, p_\perp) + h.c.) \frac{d\theta}{2} dp_\perp$$

$$m_{eff} = \sqrt{m^2 + p_\perp^2} \quad (2)$$

where we have chosen as our standard wedge the $x_1 - x_0$ wedge (invariant under the $x_1 - x_0$ boost subgroup) and parametrized the longitudinal coordinates in terms of x-space and p-space rapidities. The encoding of structural properties of the $\mathcal{A}(W)$ bulk into simpler properties of its holographic projection onto the causal horizon $\mathcal{A}(H(W))$ can then be described by passing to the limit⁴ $r \rightarrow 0$, $\chi \rightarrow \infty$ such that $x_- = 0$, $x_+ > 0$ and finite. In this limit the mass loses its physical significance and becomes a mere placeholder for keeping track of the engineering dimension

$$A_{H(W)}(x_+, x_\perp) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int (e^{im_{eff}x_+e^\theta+ip_\perp x_\perp} a^*(\theta, p_\perp) + h.c.) \frac{d\theta}{2} dp_\perp \quad (3)$$

$$\langle \partial_{x_+} A_{H(W)}(x_+, x_\perp) \partial_{x'_+} A_{H(W)}(x'_+, x'_\perp) \rangle \simeq \frac{1}{(x_+ - x'_+ + i\varepsilon)^2} \cdot \delta(x_\perp - x'_\perp)$$

$$[\partial_{x_+} A_{H(W)}(x_+, x_\perp), \partial_{x'_+} A_{H(W)}(x'_+, x'_\perp)] \simeq \delta'(x_+ - x'_+) \cdot \delta(x_\perp - x'_\perp)$$

For convenience we have taken the lightray derivatives of the generating fields; this saves us the usual ritual of restricted testfunction spaces for zero mass chiral free fields; upon *Haag-dualization* the algebras generated by the derivative fields are identical to those defined with modified testfunction smearing.

⁴Whereas the holographic projection of fields in the mass-shell representation (see below) could have been done directly by setting $x_- = 0$, algebraic holography needs the setting of modular operator theory which only works for localized algebras which fulfill the Reeh-Schlieder property.

The lightfront fields $A_{LF}(x_+, x_\perp)$ are obtained by linearly extending the $A_{H(W)}(x_+, x_\perp)$ to all values of x_+ . The fields $A_W(r, \chi; x_\perp)$ and $A_{H(W)}(x_+, x_\perp)$ (respectively their derivatives) generate operator algebras⁵ $\mathcal{A}(W)$ and $\mathcal{A}(H(W))$. The local algebras generated by the derivatives are smaller, but the extension via Haag dualization restores equality [13]. It is fairly easy to see that

$$\mathcal{A}(W) = \mathcal{A}(H(W)) \quad (4)$$

In fact by using the relation between the on-shell restriction of W-supported smearing functions and smearing on $H(W)$, the generators can be directly placed into correspondence.

This identity is surprising at first sight since $\mathcal{A}(H(W))$ or its linear extension $\mathcal{A}(LF)$ does not distinguish between a massive and a massless theory. However this identity does not extend to compactly localized subalgebras; the knowledge of lightfront generators (3) on one lightfront only does not suffice to reconstruct compactly localized algebras in the bulk. Vice versa one cannot construct the local substructure on the horizon from the bulk substructure of its associated wedge. With additional information about certain LF changing action of the Poincaré group, and by taking algebraic intersections and unions, one can however recover the local bulk structure and its pointlike generating field coordinatizations (which includes besides the free fields also its Wick-monomials). The reason behind these exact connections is that, different from the critical limit of a massive theory which leads to a conformal invariant massless theory, the Hilbert space and certain noncompact localized subalgebras are shared between the bulk and the lightfront, a fact which in our example is obvious since we never changed the Wigner particle creation/annihilation operators for our massive particles i.e. the full content of the representation of the Poincaré group remained encoded in both descriptions.

The main point of lightfront holography (as we will denote the present setting in order to distinguish it from the old lightcone quantization) is precisely a radical change of spacetime ordering while maintaining the material substrate. Since the spacetime ordering of matter is crucial for its physical interpretation, certain concepts for which the bulk description was important, as e.g. scattering theory of particles, become blurred in the holographic projection and concepts like localization entropy/energy, which are essential for the understanding of the area behavior of entropy on causal/event horizons, are not very accessible from a pure bulk point of view. Already the above free field calculation reveals that the holographic generator $A_{H(W)}$ is in many aspects simpler since it generates a transversely extended chiral theory with no transverse vacuum fluctuation. It can be used to generate $\mathcal{A}(W)$, but is of no use for generating all the subalgebras contained in $\mathcal{A}(W)$, for this one would have to use the bulk free field $A_W(x)$.

There is no direct reconstruction of the bulk field A_W from the boundary field $A_{H(W)}$, rather one has to pass through intermediate purely algebraic steps

⁵The relation between free field generators and operator algebras is defined in terms of the Weyl generators but these well-known technical points are left out in these notes.

as intersecting wedge algebras to obtain double cone algebras $\mathcal{A}(\mathcal{O})$ which for arbitrary small \mathcal{O} 's in turn lead back to the pointlike covariant fields which generate all subalgebras. Hence a constructive approach in this setting starts from the algebraic structure of a wedge algebra $\mathcal{A}(W)$ in general position (obtained from a special one by applying the Poincaré group) and the aim is to obtain the compactly localized double cone algebras $\mathcal{A}(C)$ from algebraic intersection. If these intersection algebras are trivial (scalar multiples of the identity) then there are no local observables we say that the model does not exist as local QFT.

Opposite to the standard approach which moves from pointlike fields to more extended observables, the direction of the algebraic approach is outward→inward; the local field only appears at the end in its role as the generator of algebras for all spacetime regions. These algebraic constructions have been backed up by explicit calculations which established the existence [9] of a two-dimensional class of so-called factorizing models whose S-matrix and formfactors had already been known before.

The rather direct relation between free fields and their holographic generators holds only for linear nullsurfaces i.e. the lightfront. The causal horizon of a double cone C (the intersection of two lightcones) has also chiral generators in lightray direction and vacuum-polarization-free transverse angular generators, but they cannot be obtained by a restriction procedure on free fields which generalizes the above limit leading to $A_{H(C)}(x_{H(C)})$. For conformally invariant models they can be obtained by applying the relevant conformal transformation (i.e. that one which maps the wedge into a double cone) to the $\mathcal{A}_{H(W)}$ generator [28]. Since the holographic projections are conformally invariant even if the bulk theory is not conformal, the application of conformal maps between different null-horizons even in case where the bulk is not conformal has a certain plausibility, but a rigorous mathematical justification is still missing.

The structure of the localized operator algebras in QFT are very different from those one meets in QM. Whereas the operator algebra representing the total (generally infinitely extended i.e. open) system⁶ at zero temperature in both cases is the irreducible algebra of all bounded operators in a Hilbert space $B(H)$, this does not extend to local subalgebras of QFT. Using the notation $\mathcal{N}(V) \subset B(H)$ for the von Neumann subalgebra associated to the 3-dim. region $V \subset \mathbb{R}^3$ and $V' = V \setminus \mathbb{R}^3$ its complement, the quantum mechanical tensor factorization reads

$$\begin{aligned}\mathcal{N}(\mathbb{R}^3) &= \mathcal{N}(V) \otimes \mathcal{N}(V'), \quad H = H_V \otimes H_{V'} \\ \mathcal{N}(\mathbb{R}^3) &= B(H), \quad \mathcal{N}(V) = B(H_V), \quad \mathcal{N}(V') = B(H_{V'})\end{aligned}$$

Pure states of the form $\sum \lambda_i |\psi_i\rangle \langle \psi'_i|$ which are superpositions with respect to the tensor factorization are called entangled (with respect to the tensor decomposition) and the averaging over the unobserved degrees of freedom in the say

⁶In order to have a common setting for QM and QFT one should use the Wigner-Fock multiparticle formulation of QM.

observational inaccessible region V' leads to a density matrix ρ_V for the observer confined to V . The impurity of the state can be measured in terms of the von Neumann entropy associated to ρ_V but there is no thermal manifestation of Born localization i.e. the uncertainty relation is not related to thermal behavior. *The quantum mechanical entropy resulting from "Born"-entangled states on a tensor product lead to a "cold" entropy in the sense of informations theory [15]. The localization entawking radiation.* As there is no problem of loss of information in properly understood black hole physics, the entanglement in tensor products of QM cannot be used to generate thermal manifestations.

The situation changes radically if one passes from QM to QFT [14][15] In that case a covariant localized subalgebra $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(M)$, $\mathcal{O} \subset M =$ Minkowski spacetime. In this case the complement is replaced by the causal disjoint \mathcal{O}' whose algebraic counterpart is the commutant i.e. $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$ (Haag duality, a property which can be always achieved by suitably enlarging the local algebras) and the generating and causal commuting property

$$\begin{aligned}\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\mathcal{O}') &= \mathcal{A}(M) = B(H) \\ [\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}')] &= 0\end{aligned}\tag{5}$$

In the classification of Murray and von Neumann field theoretic algebras as wedge-localized algebras $\mathcal{A}(W)$ and its causal complement $\mathcal{A}(W') = \mathcal{A}(W)'$ are still factor algebras⁷, but there is no tensor factorization and hence no prerequisite for defining the analog of the above entanglement.

In fact the local factor algebra does not admit any pure state, rather *all states are impure*. Any attempt to go ahead and ignore this difference will lead to infinities and ill-defined expressions for measuring the impurity in terms of an entropy. Although we will not enter mathematical subtleties, the basic reason for this unusual state of affairs can be traced back to the radically different nature of these local covariant subalgebras of which the most prominent example in this article is $\mathcal{A}(W)$. They are all (as long as the localization region has a nontrivial causal complement) isomorphic to the unique hyperfinite type III_1 von Neumann factor which for reasons which will become later will be shortly referred to as the *monad* in the rest of this article. It is however not our aim to enter a systematic mathematical discussion about operator algebras [16][14], rather we will pay attention to those aspects of operator algebras which are important for the problematization of holography and thermal/entropic aspects of localization [12][28].

Although, as mentioned before, monads does not arise in global zero temperature algebras $\mathcal{N}(\mathbb{R}^3)$ or $\mathcal{A}(M)$, they do however make their appearance in the thermodynamic limit of finite temperature systems. This preempts the thermal aspects of localization in QFT on the very fundamental level of single operator algebras. The vastly different behavior of localized subalgebras in QM and QFT is of course a reflection of the difference between "Born localization" [15] and modular localization [17]. The origin of the terminology "modular"

⁷Under very mild assumptions the algebras for subwedge localization regions $\mathcal{A}(\mathcal{O})$ are of the same type as $\mathcal{A}(W)$.

will become clearer in the sequel. but in the present context it stands for an intrinsic formulation (i.e. independent of the myriads of field coordinatizations which a particular model admits). "Born localization" is the standard quantum mechanical localization which is directly associated with projection operators and probabilities to find the system at a given time in a certain spatial region. In the relativistic context the Born localization is usually referred to as the Newton-Wigner localization. Its lack of local covariance makes it rather useless for the problems of causal propagation over finite spacetime distances.

But instead of pointing to its shortcomings outside QM it is more important in the context of our present discussion to stress the fact that it is asymptotically covariant in the sense of large timelike distances. Without this property (which usually goes unmentioned) there would be no covariant scattering theory with an invariant S-matrix and no associated asymptotic Born probabilities and projection operators. As Born localization goes together with the quantum mechanical type I_∞ algebras, modular localization is inexorably linked with the monad of QFT. It is fully locally covariant and permits the formulation of causal propagation over finite distances in terms of expectation values which however do not permit a further going resolution in terms of projectors. The absence of probabilities and projectors is related to the so-called Reeh-Schlieder property namely that the algebra $\mathcal{A}(\mathcal{O})$ generated by localized fields applied to the vacuum creates a dense set of states and has no vacuum annihilators.

Ignoring these structural differences produces well-known superluminal paradoxes as e.g. the alleged causality violation of Fermi's Gedankenexperiment [14]. Although not part of the present investigation, many of the recent paradoxes in connection with black holes (information paradox,...) which often are attributed to the still elusive QG probably also have their origin in an insufficient awareness about the structural differences between QM and local algebras in QFT.

Some historical remarks are in order. Vacuum polarization as a result of localization in QFT has been first noticed by Heisenberg when he attempted to study the well-known Noether connection between global conserved charges and conserved currents. The Heisenberg infinities in the vacuum fluctuations caused by the sharp spatial cutoff R can of course be avoided by suitable testfunction smoothing; in that case the total charge is the $R \rightarrow \infty$ limit of the partial charge (without changing the testfunction smoothing at the boundary).

Even more spectacular was the observation of Furry and Oppenheimer that the application of interacting pointlike fields to the vacuum Ω does not only create a one-particle state but also an unavoidable particle-antiparticle polarization cloud. In modern terminology this observation is part of a theorem which states that necessary and sufficient for a subwedge algebra⁸ $\mathcal{A}(\mathcal{O})$ to be generated by free fields is the existence of a PFG (vacuum polarization free generator, generally an unbounded operator G) affiliated with $\mathcal{A}(\mathcal{O})$ such that $G\Omega = \text{one-particle}$.

In view of these early perceptions it is a little bit surprising that the ther-

⁸This means that the causal closure can be enclosed in a wedge $\mathcal{O}'' \subset W$.

mal manifestation of modular localization came as late as 1975 [29]. It entered the general consciousness of most particle physicists only (if at all) through the thermal aspects of black hole horizons [18][19]. Sometimes these thermal manifestations are linked to the uncertainty relation but this is somewhat misleading. Whereas it is true that uncertainty relations are behind most phenomena of QM, the thermal manifestation (KMS states, entropy through localization) of field theoretic monads $\mathcal{A}(\mathcal{O}) \subset B(H)$ are outside their quantum mechanical reign. The entropy related to quantum mechanical entanglement is a "cold" i.e. information-theoretical kind of entropy whereas the localization entropy is genuinely thermal.

The rather simple relation between the generators $A_{H(W)}(x_+, x_-)$ and the original free fields appears much more involved if one asks questions about inverse holography.

The content of the subsequent sections is as follows. The third section presents the algebraic setting of lightfront holography in the presence of interactions. In case of factorizing two-dimensional models for which the relation between the S-matrix and the generators of wedge algebras and the associated chiral holographic projection becomes more explicit. The fourth section presents two quite different methods to calculate localization entropy; as expected they agree in their leading behavior in their attenuation distance for their vacuum polarization at the boundary. Some generic consequences as the area proportionality of localization entropy and the necessity to adjust the Bekenstein-Hawking black hole setting to these general structural facts of QFT before entering the more elusive terrain of QG are pointed out in the concluding remarks.

3 Lightfront holography in the presence of interactions

Since Lagrangian perturbation theory is not really suited for the investigations of holography and localization thermality, and since there are presently no reliable nonperturbative construction of models, the passing from bulk to its holographic projection presently amounts to a purely structural model-independent discussion. The free field constructions of the previous section suggests to start the holography from the position of a local operator algebra monad $\mathcal{A}(W) \subset B(H)$.

In the interacting situation there is an additional physically stronger argument which is related to the nature of the modular objects of the standard pair $(\mathcal{A}(W), \Omega)$ [7]. It is well-known that the modular group acts as the W -preserving boost whereas the anti-unitary modular inversion J depends on the interaction via the S-matrix and the free (incoming) modular inversion J_0

$$J = J_0 S_{scat} \tag{6}$$

This form of course requires the validity of a complete particle interpretation which according to S-matrix folklore is expected to follow from a mass gap. This result (which follows from the TCP invariance of the S-matrix) shows the close

connection between the S-matrix and the monad. which we want to interpret as a wedge-localized algebra. It can be strengthened by assuming the validity of crossing "symmetry" for formfactors⁹ i.e. the vacuum to n-particle matrixelements of a localized operator. In that case the inverse scattering problem can be shown to have a unique solution i.e. if there is at all a QFT associated to an S-matrix it is necessarily unique (the S-matrix is the special in-out formfactor of the unitary operator). The physical significance of the modular objects Δ^{it}, J associated to standard subwedge localized pairs $(\mathcal{A}(\mathcal{O}), \Omega)$ is not known apart from the fact that the action of the modular group is "fuzzy" i.e. cannot be encoded into a spacetime diffeomorphism. For this reason the wedge algebras play a prominent role in a new algebraic classification and construction program of interacting QFTs. Without the use of the simplifications which result from the holographic projection of the bulk localized in a wedge to its lightfront horizon such a program would not probably not be feasible. But the fact that a lot is known about the classification and construction of chiral theories generates some hope.

The important property of lightfront holography in the context of this paper is the transverse tensor-factorization for subalgebras [12][28] $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \subset LF}$ where \mathcal{O} now stands for regions on LF

$$\begin{aligned} \mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2) &= \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2), \quad (\mathcal{O}_1)_\perp \cap (\mathcal{O}_2)_\perp = \emptyset \\ \langle \Omega | \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2) | \Omega \rangle &= \langle \Omega | \mathcal{A}(\mathcal{O}_1) | \Omega \rangle \langle \Omega | \mathcal{A}(\mathcal{O}_2) | \Omega \rangle \end{aligned} \quad (7)$$

This total absence of transverse entanglement is somewhat surprising. Whereas this is a rigorous mathematical theorem, the following formulation of transverse factorization in terms of transverse extended chiral fields $B_{LF}(x_+, x_\perp)$ in analogy to the free field case (3) is only on the level of a consistency argument

$$\left[B_{LF}^{(i)}(x_+, x_\perp), B_{LF}^{(k)}(x'_+, x'_\perp) \right] \simeq \left\{ \sum_l \delta^{nl}(x_+ - x'_+) B^{(l)}(x_+, x_\perp) \right\} \delta(x_\perp - x'_\perp) \quad (8)$$

where in the case of transversely extended rational theories the algebraic structure of the theory permits a characterization in terms of a finite number of LF generating fields $B_{LF}^{(i)}$. The transverse delta functions result from transverse derivatives in the B -fields; they are associated to non-fluctuating quantum mechanical degrees of freedom since in contrast to the lightlike positive energy condition they suffer no frequency restriction and hence appear already on the level of correlation functions. This commutation relation certainly holds for Wick-monomials of free fields. Note however that unlike the free field case (3), in the presence of interactions one should not expect a factorization into longitudinal and transverse part i.e. the B 's will remain x_\perp dependent, i.e. holography leads to a (transversely) *extended chiral theory*.

⁹Rigorous (but unfortunately very elaborate) proofs of crossing only exist for formfactors with few particles. For two-dimensional factorizing models the crossing is verified as part of the explicit construction.

In the presence of interactions one expects the appearance of fields with anomalous dimensional (non-integer dimensional but bosonic) fields in the bulk. Whereas in the bulk the anomalous dimension is independent of its bosonic spacelike commutativity, in (extended) chiral theories the spin-statistics relation together with the relation between spin and scale dimension excludes bosonic observables with anomalous scale dimension. This indicates that the global algebra which the compactly localized subalgebras algebras of $\mathcal{A}(LF)$ and $\mathcal{A}(\partial W)$ generate maybe smaller than these algebras i.e.

$$\overline{\bigcup_{\mathcal{O} \subset \partial W} \mathcal{A}(\mathcal{O})} \subseteq \mathcal{A}(\partial W) \equiv \mathcal{A}(W) \quad (9)$$

We will return to this important problem in the more restricted context of $d=1+1$ factorizing models.

Whereas in higher spacetime dimensions the pointlike generating property of algebraic nets still depends on certain technical assumption, it is well-known that the simpler chiral algebraic nets always have pointlike generators [20]. Looking at the representation theoretical nature of the argument there can be no doubt that conformal nets in higher spacetime dimension also share this pointlike generator property. For massive factorizing models these fields are known in terms of infinite series whose convergence status is not yet known.

Many profound ideas of the old bootstrap of the 60s were lost as a result of its ideological hubris which led to a positioning of the S-matrix bootstrap against QFT instead of considering the use of on-shell objects and their properties (e.g. the crossing property) as a valuable enrichment of QFT. In the new context of a mass-shell based construction of QFT which tries to extend Wigner's representation theoretical approach for one-particle spaces to the realm of interactions some of these old bootstrap ideas come to new life. A pure bootstrap approach in which the classification of interacting S-matrices can be pursued separated from the construction of an associated QFT is limited to two spacetime dimensions; in higher dimensions the S-matrix plays the role of a special formfactor and the formfactor program in turn becomes incorporated into the construction of generators for wedge algebras.

4 The special case of factorizing models and their holographic projection onto a chiral QFT

Many profound ideas of the 60s, which were lost in the aftermath of the "TOE hubris" when the S-matrix bootstrap program was positioned against instead of within QFT, were later on vindicated when it was observed¹⁰ that there is a rich class of interacting two-dimensional massive QFT which are uniquely associated with bootstrap S-matrices. The first observation which led to a kind of

¹⁰It was based on the integrability features observed in quasiclassical approximation of the particle spectrum in certain two-dimensional QFTs by Dashen, Hasslacher and Neveu which in case of the Sine-Gordon theory was afterwards explained in terms of the bootstrap S-matrix properties restricted to two-particle elastic scattering [21].

revival within a more limited context of integrable QFT was about the possible integrability of certain two-dimensional Lagrangian QFT on the basis of their quasiclassical mass spectrum. This was followed by the remark that in case of the Sine-Gordon model the mass formula follows from an exact computation based on applying the bootstrap requirements to an elastic two-particle S-matrix Ansatz [21]. In two dimensions it is consistent with macro-causality (in particular with the cluster factorization property) to have a purely elastic S-operator which factorize in terms of an elastic two-particle scattering amplitude; the associated QFTs were therefore named *factorizing models*. These S-matrices were susceptible to a systematic classification obtained from the bootstrap principles: unitarity, Poincaré invariance and the crossing property [6]. This bootstrap S-matrix setting was then connected via the bootstrap-formfactor program with a new way of nonperturbative construction of factorizing QFTs [22].

Different from the original hope that the bootstrap principles would lead to unique TOE¹¹, it turned out that the factorizing setting rather led to a extraordinary rich family of infinitely many nontrivial two-dimensional nonperturbatively controllable models. Besides those models which permit a Lagrangian description (in the sense of a Lagrangian "baptism" and a divergent perturbative series rather than a construction), there is an enormous number of non-Lagrangian models (the oldest and most prominent being the $Z(N)$ model [23]) which is a consequence of the fact that there are more elastic two-particle bootstrap S-matrices $S^{(2)}$ (i.e. more solution of the Yang-Baxter equations) than interactions \mathcal{L}_{int} . There is no physical principle which distinguishes Lagrangian QFT relative to those of non-Lagrangian origin since the quantization parallelism to classical physics is not a physical principle and there is no intrinsic autonomous property of QFT which is capable to reveal a Lagrangian origin.

The next step was to encode the elastic S-matrix data into an algebraic structure. The resulting Zamolodchikov-Faddeev (Z-F) algebra has the appearance of a non-local generalization of the Wigner momentum space creation/annihilation operators into which the S-matrix data enter as structure coefficients which characterize the commutation structure (11). The observation that the on-shell Fourier transforms into x-space yields a covariant on-shell generator of a wedge-localized algebra bestows a *physical interpretation in terms of spacetime localization* to these initially purely auxiliary nonlocal operators [7]. Modular localization and the related Tomita-Takesaki modular operator theory of operator algebras plays a crucial role in obtaining these results.

As mentioned in the previous section modular theory also shows that, contrary to the old bootstrap philosophy, the S-matrix of a QFT is not completely void of spacetime localization aspects. Rather it is deeply connected with wedge-localization in the sense that S is a *relative modular invariant* of the wedge-algebra (6). Considered as a in-out formfactor of the unit operator, and combined with the formfactors of all other operators, the S-matrix leads to the *uniqueness* of an associated QFT (if it exists) *if the formfactors fulfill the cross-*

¹¹Apart from gravity which was already missing in the Heisenberg's "Weltformel", probably the first futile attempt at a TOE before this strange ideas like that became fashionable in particle physics.

ing property [24]. In general crossing only assures the uniqueness of the inverse scattering problem; no argument is presently known which secures its existence in a general setting including crossing.

Crossing is a deep analytic on-shell property which is not valid in other non QFT based relativistic S-matrix theories which implement interactions directly (without the mediation of QFT) in a Wigner multiparticle representation theoretical setting, the so-called *direct particle interactions* [25][15]. Hence crossing may be considered as the on-shell imprint of the field theoretic locality and spectral properties in conjunction with the assumption of completeness of asymptotic particle configurations. In the nonperturbative setting of QFT it was only proven for special particle configurations; but since in factorizing theories crossing is part of the construction, the existence proof for those models also shows the validity of this property within this class of factorizing QFTs.

The setting of two-dimensional factorizing models is presently the only known case in which the bootstrap-formfactor conjecture (S-matrix, formfactor) \rightarrow QFT can be backed up by constructive mathematical steps. This class of factorizing models is not only interacting in the global sense of a nontrivial S-matrix, but it also permits a local characterization in the sense of possessing no subwedge localized $\mathcal{A}(\mathcal{O})$ -affiliated PFGs (vacuum-polarization-free-generators¹²). The different type of interactions in the latter case would instead of corresponding to different Lagrangians be characterized by different S-matrices (different wedge generators) or on the local level by different shapes of the local vacuum polarization clouds (still somewhat futuristic). These properties are expected to continue to hold also in the general case when there are no so called *temperated wedge-localized PFGs* (see below).

The characteristic feature of factorizing models, which entails the relative simplicity of their construction, is that they possess *no real (on-shell) particle production*. But the characteristic feature of interacting QFT is not particle production (which can also be incorporated into *direct particle theories* [25]) but rather the *interaction-caused infinite vacuum polarization clouds* which result from compact spacetime localization. This is in contrast to the localized interaction-free algebras which always admit PFGs leading to the existence of underlying free field generators for arbitrary localizations and for which the composite fields (Wick-powers) generate vacuum polarization with only a finite number of particle/anti-particle pairs (the number depending on the degree of the Wick monomial). The existence of temperate PFGs for wedge regions only maintains the on-shell wedge generators close to their free field form; they obey the slightly more general Z-F commutation relations which lead to the factorization of the S-matrix which is the origin of the terminology for this class of models.

A systematic structural study of PFGs in the presence of interactions in general QFT was initiated in [8] and it was found that whereas for compact localization regions any operator has the infinite vacuum polarization cloud

¹²PFGs are operators which applied to the vacuum generate a one-particle state with admixture of vacuum polarization clouds. Only for free fields they exist for compact localization regions.

(confirming previous results), it is only for the noncompact wedge region where modular theory leads to the existence of PFG.

Unfortunately modular theory does not guaranty reasonable domain properties which permit a successive application of these unbounded operators as for smeared Wightman fields. Only if these wedge-localized PFGs are *temperate* [8] in the sense of good domain properties one has been able to extract interesting consequences from their existence. According to an old theorem [5] this is impossible in higher dimensions; using some rigorous analytic properties for the scattering amplitude one finds that this is only possible in $d=1+1$ [8]. With the help of crossing for formfactors one can exclude higher direct elastic multiparticle amplitudes so that the higher multiparticle scattering has to go through $S^{(2)}$ which is the S-matrix definition of factorizing QFTs.

So the notion of integrable or factorizable theories can be fully substituted by the notion of *theories with temperate PFG's*. With this change of paradigmatic emphasis, QFT returns to its beginnings when Furry and Oppenheimer discovered (perturbatively) the omnipresence of interaction-caused vacuum polarization clouds; but now with rich additional conceptual additions from modular localization theory and the hope that perturbative Lagrangian quantization can be replaced by characterizing interacting theories in terms of wedge generators and structural properties of subwedge polarization as it is presently happening in factorizing models.

Modular operator theory applied to the wedge-localized algebra of QFT $\mathcal{A}(W)$ leads to a new semilocal interpretation of S-matrices: as mentioned before (6) they are *relative modular invariants* (relative to the free wedge algebra $\mathcal{A}_0(W)$ without interactions). This important insight which links scattering to fields (i.e. part of the inverse scattering program) was missing in the old S-matrix bootstrap approach. It confirms that the S-matrix preempts an important semiinfinite localization aspect of QFT and that it was not wise to position the S-matrix bootstrap program *against* QFT not to mention the aborted attempt to convert it into a TOE.

The for the time-being last step in this interesting sequence of events is Lechner's recent existence- and particle completeness- proof [9] in the setting of factorizing models. It is based on the phase-space property of *modular nuclearity*; this result in some way vindicates the four decades old Haag-Swieca paper on the connection of local fields and asymptotic particles via phase space properties by giving a positive answer to the question in the title of their paper. It also shows that particle physics does not only consists of changing fashions but that there are once in a while ideas with a historical breath.

Let us sketch some details of these constructions. The starting point is the formula for the wedge generators in terms of the Z-F algebra operators. In the scalar one-component case these generators have a form similar to free field, except that the on-shell creation/annihilation operators which appear in their Fourier transform fulfill the Z-F commutation relation¹³ (which are slightly more

¹³The $\tilde{Z}^\#(\theta)$ in general are multi-component and the *-algebra requirement forces the matrix-valued structure functions $S(\theta)$ algebra to be unitary solutions of the Yang-Baxter

general than the Wigner creation/annihilation operators).

$$Z(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int (e^{ip(\theta)x} \tilde{Z}(\theta) + h.c) \frac{d\theta}{2}, \quad p(\theta) = m(ch\theta, sh\theta) \quad (10)$$

$$\begin{aligned} \tilde{Z}(\theta) \tilde{Z}^*(\theta') &= S(\theta - \theta') \tilde{Z}^*(\theta') \tilde{Z}(\theta) + \delta(\theta - \theta') \\ \tilde{Z}(\theta) \tilde{Z}(\theta') &= S(\theta - \theta') \tilde{Z}(\theta') \tilde{Z}(\theta) \end{aligned} \quad (11)$$

This has the consequence that although the $\tilde{Z}^\#$ commutation relations remain close to those of the Wigner-Fock creation/annihilation operators and the field $Z(x)$ transforms like a would be pointlike field under the Poincaré group, the $Z(x)$ is *not pointlike local* in the sense of spacelike commutation relations. It also shows a covariant coordinate label x is not necessarily indicating pointlike localization, although in the opposite direction the statement is correct.

On the other hand $Z(x)$ is *not completely nonlocal* either; since the application of modular operator theory reveals that operators $Z(f)$ with $\text{supp} f \in W$ are only W -local (*semilocal*), no matter how sharply one localizes the support inside W . The exponentiation of unbounded smeared $Z(f)$ operators leads to the well known Weyl algebra structure whose weak closure (or double commutant) defines the operator algebra $\mathcal{A}(W)$ which according to modular theory is W -localized.

In this way the Z-F operators, which at first appeared as purely formal auxiliary objects in the formfactor-bootstrap program, are given a spacetime interpretation. They are objects which have better relative locality properties (always relative to the pointlike interacting fields) than the in-out free fields. Instead of being completely relatively nonlocal they at least are relative wedge local with respect to the interacting fields and with respect to themselves and hence fully wedge local. The nonlocality (better semilocality) of $Z(x)$ is the prize to pay for the absence of the vacuum polarization in the vector-valued distribution $Z(x)\Omega$.

It is not our aim here to study nonlocal theories for their own sake; it is the fact that they are *semilocal in the sense of wedges* which makes the $Z(x)$ very interesting as intermediate objects on the way to genuine local theories.

This has an interesting connection with a kind of the particle-field complementarity in the presence of interactions¹⁴. In a free theory one-particle states can be generated from the vacuum by the application of smeared fields with smearing functions of arbitrary small support (PFG's exist for arbitrary small localization regions). The presence of any interaction radically changes this state of affairs [15]: there is no relative compact region for which the PFG property prevails and the smallest noncompact causally complete region (causal completion is automatic in the algebraic approach) for which PFGs prevail even

equations. It follows in the course of construction of the theory that the S-coefficients of the algebra are also the two-particle matrixelements of the scattering matrix (showing again the close relation of the scattering matrix to local aspects of QFT).

¹⁴It should not be confused with Bohr's particle-wave complementarity which persists without interaction.

in the presence of interactions are the wedges. In this sense the *wedge region leads to the best compromise between particles and fields*.

Particles are not only important in scattering theory¹⁵ but they play the crucial role in nonperturbative construction of models via the bootstrap-formfactor construction of factorizing models and the relative modular invariance property for wedges of the S-matrix. In contradistinction to quantum fields which coordinatize algebras (similar to the role of coordinates in the modern formulation of differential geometry) and are therefore not of *direct* physical significance, *particles have an ontological individuality/objectivity* [15]. But in the presence of interactions they are in the above sense nonlocal, and therefore it is not surprising that nonlocal intermediate steps are helpful in nonperturbative constructions.

Schematically one proceeds as

$$\text{temperate PFG for wedge} \longrightarrow ZF\text{-algebra} \longrightarrow \mathcal{A}(W) \quad (12)$$

One-particle states free of vacuum-polarization can always be created from the vacuum by the application of unbounded wedge-localized PFGs [8], which permits a particle interpretation. But in but in order to be able to utilize them for constructive purposes one must presently require that they be "temperate", i.e. their range is tuned to their domain in such a way that an iterative applications (as in Wightman field theory) is possible. It is precisely this requirement which forces the restriction of factorizability and therefore the two-dimensionality QFT on us and it is only in this special setting that the bootstrap classification and computation of S-matrices can be separated from the formfactor construction which requires the setting of QFT.

In view of the fact that historically the first investigations of factorizing models proceeded through the quantization of classically integrable field models, and in view of the complicated nature of classical integrability (infinitely many conservation laws), it comes as a pleasant surprise that a simple restriction in terms of vacuum polarization leads to the same result in a purely intrinsic QFT way. It is one of several known instances in which quantum arguments are conceptually simpler than their classical counterparts.

The construction of the wedge algebra $\mathcal{A}(W)$ from the Z -generators is entirely analogous to the construction of the Weyl algebra from the free fields. Whereas on the level of the S-matrix and the wedge generators of the form 10 the theory has the appearance of a relativistic potential theory, this state of affairs changes radically if one passes to compact localizations as the Poincaré covariant family of double cone algebras $\mathcal{A}(\mathcal{C})$ which arise as relative commutants of wedge algebras

$$\mathcal{A}(\mathcal{C}) = \bigcap_{W \supset \mathcal{C}} \mathcal{A}(W) \quad (13)$$

¹⁵Scattering theory is build on the idea that multiparticle states asymptotically stabilize: if at asymptotically large times by counter coincidence and anticoincidence arrangements one established the presence of an precisely n-fold localized state then it remains this way.

where in the two-dimensional case one only needs to intersect the standard wedge (with apex at the origin) with its translated opposite. Intersecting algebras is a task for which presently no tools exist. However, as a result of the simplicity of the algebraic structure of factorizing models, there are two ways to do this. The more formal procedure starts from a general Ansatz ($p(\theta) = m(ch\theta, sh\theta)$)

$$A(x) = \sum \frac{1}{n!} \int_C d\theta_1 \dots \int_C d\theta_n e^{-ix \sum p(\theta_i)} a(\theta_1, \dots, \theta_n) \tilde{Z}(\theta_1) \dots \tilde{Z}(\theta_n) \quad (14)$$

where for reasons of a compact notation we view the creation part $\tilde{Z}^*(\theta)$ as the $\tilde{Z}(\theta + i\pi)$ i.e. as the Z on the upper boundary of a strip¹⁶ (we could have introduced this notation already in (10)).

This is similar to the GLZ representation of the interacting Heisenberg field in terms of incoming free field, in which case the spacetime dependent coefficient functions turn out to be on-shell restrictions of Fourier transforms of retarded functions except that instead of the on-shell incoming fields one takes the on-shell Z operators which conceptually are somewhere between Heisenberg and incoming fields.

In the latter case the coefficient functions are precisely those formfactors which feature in the bootstrap-formfactor approach to factorizing theories. Together with a certain analytic requirement on the coefficient functions, the space of these formal power series represent the space of formal (in the spirit of vertex operators) \mathcal{W} -localized fields. Taking for \mathcal{O} a double cone \mathcal{D} whose left apex coalesces with the origin and representing \mathcal{D} as the intersection of the standard right wedge with an $a > 0$ translated standard left wedge, the calculation of $\mathcal{A}(\mathcal{D})$ -generating operators is based on the relative commutant restriction placed on the coefficient functions:

$$[A(x), U(a)U(j)Z(f)U(j)^*U(a)^*] = 0 \quad (15)$$

In words: the generators of the right wedge (the A 's whose coefficient functions have the correct θ -strip analyticity corresponding to the right wedge localization) are subjected to the restriction that only those which commute with the generating Z 's of the a -shifted opposite wedge are admitted (i.e. the ones which generate $\mathcal{A}(\mathcal{D})$). Here $U(j)$ is the free TCP operator i.e. the one which acts on the multiparticle wave functions in the standard way and therefore $Z^{opp}(f) = U(j)Z(f)U(j)^*$ are generators of the opposite wedge. Since the commutation with the restricting operators Z_a^{opp} map the n^{th} order term in A with the adjacent $n+1$ and $n-1$, one obtains a rather simple linear recursion for the coefficient functions.

In praxis one uses this commutation relation together with covariance in order to construct a basis of composite fields within each superselection sector. Formally the space of generating operators¹⁷ for compactly localized operators

¹⁶The notation is suggested by the the strip analyticity coming from wedge localization. Of course only functions but not field operators or their Fourier transforms can be analytic.

¹⁷For unbounded operators associated to algebras (of bounded operators) it is more appropriate to speak about spaces than algebras if one has not said anything about dense domains.

is given in terms of infinite series in the Z operators with coefficient functions which obey the same relations which are known as "the formfactor axioms" in the bootstrap formfactor approach [26]. One obtains an infinite space of field generators in terms of the infinite space of formfactors in the bootstrap-formfactor program.

As in the case of free fields and their Wick composites one has "basic" (there is no Lagrangian hierarchy here) fields which by definition are those pointlike fields which if together successively applied to the vacuum generate the Hilbert space (they act cyclically on the vacuum) and the remainder are composites. The latter are expected to look like classical local monomials in the basic fields except that there is a spacetime limiting *normal order prescription*. In the Z -expansion it is easy to see that the composites share with the basic fields the nucleus of the formfactor construction (minimal formfactor) and deviate only in certain momentum space polynomials, but to translate these observations into normal product formulas for composites is not possible within the present state of QFT technology.

Since attempts to show convergence of (14) have failed¹⁸, it is deeply satisfying that there is at least an existence proof for nontrivial intersections of wedge localized algebras based on phase-space behavior ("nuclear modularity") [9] which allows to bypass the convergence problem of such series representations. This shows that the algebraic setting is not only a valuable conceptual field-coordinatization-free guide to the get to the right starting point for doing calculations in terms of field coordinates (which is the way we used it), but that it is also capable to shed some new light on age old problems of QFT, as the problem of their existence beyond formal perturbative power series with the unresolved convergence status.

The on-shell representation of Heisenberg fields (10) as an infinite series is a particularly useful starting point of the holographic projection since apart from the convergence problem it is mathematically and conceptually less demanding. It avoids the use of the still somewhat unfamiliar modular theory and uses the more standard apparatus of QFT. Instead of aiming at rigorous proofs it satisfies itself with consistency checks. and delegates the more ambitious existence proofs based on modular theory to a second stage of mathematical refinement.

With the help of the infinite series expansion (14) we can proceed along the lines of a naive restriction argument (restricting plane wave factors to a lightfront) as was done for the free field by using its on-shell representation. But there is one stain which should not be suppressed In order to arrive at on-shell formulas one has to go through the nonlocal steps of scattering theory. Hence the use of such on-shell formulas is somewhat against the spirit of simplification of certain properties (as the short-distance behavior) through lightfront holography *before* starting any calculation.

Let us take two well studied models and extract some interesting informa-

¹⁸It is perfectly consistent (with everything which one knows about divergence of perturbative series) that these series diverge; since each single term is analytic in a small circle around zero coupling, this would put the blame of divergence of perturbative series of off-shell objects on vacuum polarization.

tions from their two-point function using their holographic series representation. According to the previous remarks, the general formfactor series representation for the holographic two-point function reads (x_+ translational lightray variable, $p_-(\theta) = e^{-\theta}$)

$$w(x_+) = 1 + \sum \frac{1}{n!} \int d\theta_1 \dots \int d\theta_n e^{-ix_+ \sum p_-(\theta_i)} b(\theta_1, \dots, \theta_n) \quad (16)$$

$$= \exp \sum \frac{1}{n!} \int d\theta_1 \dots \int d\theta_n e^{-ix_+ \sum p_-(\theta_i)} b_c(\theta_1, \dots, \theta_n) \quad (17)$$

$$b(\theta_1, \dots, \theta_n) = |a(\theta_1, \dots, \theta_n)|^2$$

where in the second line we have used the Ursell-Mayer expansion which expresses the coefficient functions in terms of their cumulants b_c . Obviously the $\ln w$ series is more convenient if we are interested in the anomalous dimension of the holographically projected field.

From this series one may read off the anomalous dimension of the respective field. Apart from the critical limit in the work of Babujian and Karowski [23] which in the holographic approach is replaced by an exact bulk-boundary relation (and not by another bulk theory in the same universality class), we can take over all their formulas, in particular the formula for the dimension d_A of a algebra-affiliated field $A(x)$

$$w(x_+) = \text{const} (x_+)^{-2d_A} \quad (18)$$

$$d_A = \frac{1}{2} \sum \frac{1}{n!} \int d\theta_1 \dots \int d\theta_{n-1} b_c(\theta_1, \dots, \theta_{n-1}, 0)$$

For the Ising field theory one can do all the integrals as in [23] and then sum the series in order to obtain the expected result $d_A = \frac{1}{16}$. For the Sinh-Gordon model the contribution to the series up $n=2$, the authors arrive at a rather complicated function in terms of the Sinh-Gordon coupling strength whose further evaluation has to be done numerically. Holography with pointlike fields leads to the same integrals, its only advantage is that its relation to the bulk is exact since it does not change the algebraic substrate but only its spacetime ordering.

The main difference to the present derivation is conceptual; whereas Babujian and Karowski go to the critical limit which is associated with a massless QFT associated with different operators which act in a separate Hilbert space, holography takes place in the same Hilbert space and certain wedge-localized algebras whose upper causal horizon lies on the lightfront are shared between bulk and holographic projection. The set of shared algebras is invariant under a certain 2-parametric subgroup of the 3-parametric Poincaré group \mathcal{P} and in order to re-construct the mass spectrum one must know how the missing Poincaré transformation (e.g. the opposite lightray translation) acts on these shared subalgebras. The action on LF is necessarily nonlocal (fuzzy) i.e. it cannot be described in terms of geometry.

On the other hand the holographic projection acquires a new symmetry whose presence was not noticed in the bulk description, namely the Moebius

rotation which together with the two transformations inherited from the bulk constitutes the 3-parametric Moebius group $SL(2, R)$. The reason why it was not noticed in the bulk is because its action on the bulk is "fuzzy"; only on the horizon it becomes geometric. The concept of algebraic transformations of the bulk which become which are not related to Noether's theorem and become only geometric upon restriction is a new not yet explored structure of QFT.

In fact one expects the holographic projection to have the covariance under the full diffeomorphism group $Diff(S^1)$, even though it does not arise from a chiral decomposition of two-dimensional conformal QFT. The beauty of factorizing models is that as a result of the presence of the Z-F algebra one can study all these questions in a reasonably controllable setting i.e. *factorizing models are presently the best theoretical laboratory for testing conjectures* beyond perturbation theory.

Among the ideas waiting for a test is the conjecture that not only the free field holography for which the Diff-invariance is an obvious consequence of the transverse-longitudinal factorization of the two-point function (3), but also the holographic projections of factorizing models are automatically Moebius invariant and (under mild additional restrictions) even $Diff(S^1)$ -covariant. In higher dimensions the invariance group is expected to be even larger in the sense that the algebraic holographic structure also allows certain x_\perp -dependent chiral diffeomorphisms which are automorphisms of the algebraic commutation structure of extended chiral theories.

The fact that the holographic projection has more symmetries than those of the invariance group of the lightfront has been called symmetry enhancement on the horizon [27]. In the case of the Moebius rotation within factorizing models it means in particular that the rotation generator L_0 can be written as an infinite series in the Z s, whereas the translation and dilation retain their usual bilinear form. Besides the commutation relations the L_0 is restricted by the requirement that the vacuum is annihilated. Since the one particle creation in the bulk loses its physical meaning in the holographic projection, the application of L_0 to the one-Z state $Z(x)\Omega$ adds infinitely many Z-"quanta". In other words the Z-description is not a very natural basis if used within in a chiral theory since it has these unusual aspects.

Another important structural problem which still awaits clarification is the question how the more rigorous algebraic holography (for details we refer to [12][28][15]) is related to the holographic projection in terms of pointlike fields. The obvious conjecture in case of factorizing models is that the holographic bosonic observable algebras are generated by the holographic projection from bulk field which in addition of being bosonic also have integer short distance dimension. Here the free field is atypical because in that case all composites have integer dimension and there are no bulk fields with anomalous which survive the algebraic holography process which only passes the those operators which are bosonic in the sense of the lightray and which therefore must have integer scale dimension on the lightray.

On the other hand the holographic projection in the sense of pointlike fields does not suffer these restrictions to integer short distance behavior in the bulk,

but those anomalous dimensional bulk fields will lose their bosonic spacelike commutation structure upon holographic restriction and have braid-group commutation relations on the lightray. So in case of algebraic holography for factorizing theories there seems to be no alternative than to reconstruct the missing plektonic fields via the DHR superselection theory.

Most of the statements and conjectures, except those involving Z s can be formulated in higher dimension. The higher the spacetime dimension, the more lightfront changing transformation one must apply in order to recover the local structure of the bulk from that of the lightfront by inverse holography.

The holographic projection is an excellent method for calculating properties which are caused by the spacetime localization of quantum matter as e.g. the entropy of localization. Since this entropy results from the infinite vacuum polarization cloud on the boundary of localization, it is not necessary to know details about the localization substructure inside the bulk. This legitimizes to perform entropy calculations in the holographic projection i.e. to reduce the calculation of localization entropy for the wedge algebra to that for a semi-line which is conformally equivalent to an interval.

We know since Heisenberg's times that the vacuum polarization of sharply localized relativistic matter is infinite and therefore we have to attenuate¹⁹ those particle/anti-particle pairs by a "split procedure" (see next section) which requires to approximate the interval from the inside by a sequence of smaller intervals. Conceptually this is not much different from the formation of the thermodynamic limit for a heat-bath thermal theory. The prerequisite for this relation is that the global algebra in the heat bath representation defined by a KMS state and the global algebra in the vacuum representation after restriction to a localized subalgebra are of identical type. This is the case since both algebras are of the same type, they are what we called a *monad* in [15] As a result of the conformal invariance after holographic projection, even the geometric description becomes conformally equivalent; the infinite volume factor (i.e. infinite length l) of the in the holographic lightray theory is to be replaced by the logarithm of a diverging invariant ε which one can form from 4 points and which goes to zero as the shortest distance of the endpoint of the smaller to those of the bigger interval the distance i.e. diverges as $|\ln \varepsilon|$ The result which will be derived in the next section.

There is one more reason why the holographic projection is the preferred method for dealing with bulk properties in particular in the case of factorizing models. The Z -generators (10) of the half-lightray are the same as those for the wedge, except that the plane wave factors are those of a one-dimensional QFT. Instead of determining operator algebras associated with intervals on lightlike lines via algebraic intersections and derive the Moebius covariance via modular theory, one can also try to find a formula for the *Moebius rotation in terms of*

¹⁹In the case of "partial" charges which are formally obtained by integrating the zero component of a quantum current over the volume V this is done by smearing with a test function which goes to zero smoothly within a finite "attenuation collar" which is attached to the volume. In the infinite volume limit the dependence on the smearing function drops out and one obtains the global charge.

a series in the Z-operators.

One expects that the Moebius invariance continues to be valid beyond the holographic projection of the free theory. The convergence of the infinite series which represents the anomalous dimension (17) of the holographic projection of the bulk disorder field in the massive Ising model to the correct value for the chiral Ising model is an encouraging consistency check.

The formal arguments in favour of the Moebius covariance of holography are on the same level of rigor as the "proof" of dilation invariance of the zero mass limit, but here we want to stay close to the spirit of mathematical physics where consistency checks are not sufficient.

It is very important to understand these connections between bulk and its holography, and the factorizing models provide presently the best theoretical testing ground. For people who know chiral QFT via the standard approach, it is highly surprising that such theories (at least in those cases where they arise via holographic projections) have another (in addition to the L_0 Fourier decomposition) particle-like description in terms of a non-Moebius covariant Z-system (it lacks rotational Moebius covariance since the presence of Z-polarization clouds causes a complicated transformation property under Moebius transformations).

Whereas several results in this section depend on the factorizability of the model, the idea that the structure of the wedge algebra should form the central spine of a new completely intrinsic constructive approach to QFT is generic. Naturally nobody with any experience in particle physics would expect that outside of factorizing models one can calculate an S-matrix exactly using only the bootstrap prescription. Since the S-matrix in the present setting is the formfactor of the identity operator, one should rather view the determination of the S-matrix as part of the formfactor program where all formfactors must be determined together. The crucial hint comes from modular theory which relates the S-matrix to the so-called modular inversion (6) which coalesces (apart from a spatial rotation) with the TCP operator.

This permits to think about an onshell perturbative approach for formfactors in which the interaction input is not a Lagrangian but rather a lowest order S-matrix. Since in such an approach there is no place for singular pointlike fields but only for generators of wedge algebras, one does not expect new parameters arising from renormalization. Hence in such a still futuristic perturbative setting there should be many more finite parametric models invariant under renormalization group transformation than in the pointlike Lagrangian renormalization approach. Such an explosion of new finite parametric models is already evident in the factorizing situation where e.g. the infinitely many Sinh-Gordon type S-matrices which one obtains via CDD pole modifications all have uniquely related QFT but no Lagrangian name [9].

A very interesting soluble theory (as a result of its exotic statistics and crossing relation) is the $Z(N)$ model. It derives its name not from a Lagrangian (In fact for $N > 2$ it is not expected to have any). but rather because it was defined by the requirement that its S-matrix should implement the idea: antiparticle = bound state of $N-1$ particles. Which is the minimal way of implementing nuclear democracy within $Z(N)$ symmetry. There is no reason to believe that QFT is

tight to "baptizations" in higher dimensions.

5 The area density of localization-entropy via the inverse Unruh effect

After having established the d-2 dim. area proportionality of localization entropy, the remaining task is to use the rather detailed knowledge about chiral theories in order to calculate the dependence of this area density on the variable attenuation size ε of the vacuum polarization cloud.

There are two quite different ways to achieve this. One is based on a kind of *inverse Unruh effect* for chiral theories: the monad $\mathcal{A}(0, \infty)$ with respect to the vacuum is unitarily equivalent (via a conformal map) to a KMS state at $T = 2\pi$ on the global algebra $\mathcal{A}(-\infty, +\infty)$, in terms of the standard pair notation (the halfcircle after the compactification $\mathbb{R} = S^1$)

$$(\mathcal{A}(0, \infty), \Omega) \simeq (\mathcal{A}(-\infty, +\infty), \Omega_{2\pi}) \quad (19)$$

This conformal equivalence has a generalization to the restriction of the vacuum to chiral algebras $\mathcal{A}(a, b)$ localized in arbitrary intervals; in this case the temperature changes with the interval.

One expects the energy and entropy of the right hand side to have the usual (one-dimensional) volume proportionality i.e. $s = l s_{2\pi}$ where l corresponds to the standard volume factor and $s_{2\pi}$ to the volume density. The unitary equivalence map intertwines the translation of the heat bath theory on the right hand side with the dilation on the left hand side. In particular it transforms the length l into $\varepsilon = e^{-l}$ so that the area density in question behaves as²⁰

$$s_{area} = |\ln \varepsilon| s_{2\pi} + \text{finite}, \quad \varepsilon \rightarrow 0 \quad (20)$$

The remaining problem consists in verifying the l -proportionality and computing the coefficient $s_{2\pi}$ in its dependence on the data of the chiral model. This is achieved by approximating the divergent entropy of the heat bath system by the high temperature limit of a rotational system where the temperature is interpreted as a radius whose size is related to l). This "*relativistic box quantization*", which constitutes the second step, holds also in higher dimensional conformal QFTs [31]. The last crucial step consists in using the *temperature duality* which holds for the rotational partition function of $\hat{L}_0 = L_0 - \frac{c}{24}$. In this way one verifies the l -proportionality and finds $s_{2\pi} = \frac{c}{12}$. The three steps have been described in more detail in [12][28].

A more refined formulation of the split process in which the localization entropy of a chiral interval (a, b) is approximated from the inside by (c, d) relates ε with the conformally invariant cross ratio

$$\varepsilon^2 = \frac{(b-a)(d-c)}{(c-a)(b-d)} \quad (21)$$

²⁰ A more refined analysis reveals that the attenuation length ε is really a short hand notation for a unharmonic conformally invariant ratio.

This conformally invariant dependence instead of the volume factor could have been introduced as a conformal refinement for the l dependence already for the chiral heat bath entropy.

Note that whereas the above "inverse Unruh effect" as well as the temperature duality is not expected to hold beyond chiral theories, the "relativistic box" approximation of the heat bath thermodynamic limit is well-defined in every in every conformally invariant theory independent of spacetime dimensions.

With the insight that chiral localization entropy is equal to heat bath entropy apart from a change in the parametrization resulting from the conformal equivalence, the holographic localization entropy and its universal area proportionality has been considerably demystified. The main open problem in the application to black holes is to understand whether and how quantum gravitation is capable to lead to a numerical value for ε ; according to its microscopic derivation all values of ε are consistent with the Hawking's thermal radiation. Arguments that the value can be obtained by thermal re-interpretation of a classical area density are still frail; the preservation of a classical value in the quantum setting would appear totally unusual. Since the realistic derivation of the Hawking radiation of a collapsing star cannot be done in a thermal equilibrium setting but rather involves a stationary entropy flow, one may question the applicability of all thermal equilibrium ideas (including the present one) to black hole physics.

Although the relativistic box approximation is a conformal improvement of the standard box approximation in the formulation of the thermodynamic limit, it is desirable to have a more intrinsic formulation in which the thermodynamic limit is approached by a sequence of genuine subsystems (Boxes are belonging to unitary inequivalent systems which are only subsystems in a metaphorical sense). This will be done in the next section which does not use any of the three previous facts but is solely based on the split property. In this way one is able work with a definition of localization entropy which in principle is capable to describe the dependence on the attenuation cloud for finite ε and not only the leading terms (in the heat bath case the box quantization is only trustworthy in its leading volume term).

6 Localization entropy via the split density matrix

The second approach to localization entropy also draws its strength from chiral simplifications, but instead of conformally connecting the localization thermality of a chiral system to its heat bath KMS properties via the somewhat metaphoric "relativistic box approximation" of the previous section addresses it makes direct use of the split property which identifies the approximating algebra as a bona fide subalgebra of the same mathematical description.

In the algebraic setting a QFT is fixed in terms of a space-time indexed net of operator algebras. In the context of a chiral theory this means the net of

operator algebras indexed by proper intervals I on a circle $S^1 \simeq \mathbb{R}$ where we will use the \mathbb{R} setting of the one-point compactified line. We pick 4 points on the line $b_1 < a_1 < a_2 < b_2$ and consider the algebras $\mathcal{A}(I_a) \subset \mathcal{A}(I_b)$ where $I_a = (a_1, a_2)$, $I_b = (b_1, b_2)$ are properly included intervals. Under rather mild assumptions about phase-space degree of freedoms which are certainly valid in chiral models with a finite partition function $Z = \text{tr} e^{-\tau \tilde{L}_0}$ the split property (as studied in the second section) is valid and leads to the following tensor factorization

$$\begin{aligned} \mathcal{A}(I_a) \vee \mathcal{A}(I'_b) &\simeq \mathcal{A}(I_a) \otimes \mathcal{A}(I'_b) \\ B(H) &= \mathcal{N} \otimes \mathcal{N}', \quad \mathcal{A}(I_b) \subset \mathcal{N} \subset \mathcal{A}(I_b) \\ V(\mathcal{N})\mathcal{A}(I_a) \vee \mathcal{A}(I'_b)V(\mathcal{N})^* &= \mathcal{A}(I_a) \otimes \mathcal{A}(I'_b) \end{aligned} \tag{22}$$

Here I'_b denotes the complement of I_b and we used Haag duality $\mathcal{A}(I_b)' = \mathcal{A}(I'_b)$. To every concrete split i.e. the existence of an intermediate quantum mechanical type I factor between two monads $\mathcal{A}(I_b) \subset \mathcal{N} \subset \mathcal{A}(I_b)$ there exists a unique (by suitable normalization) implementer $V(\mathcal{N})$ of the split isomorphism.

The many different splittings correspond vaguely to classical boundary conditions, but as a result of the increase of possibilities caused by the finite thickness $a_1 - b_1$ and $b_2 - a_2$ of the two boundary between I_a and I'_b there are vastly more possibilities than in the classical case, although one expects (as for the heat bath systems in the thermodynamic limit) that they share the leading $\ln \varepsilon$ behavior.

Mathematically there is one preferred split in which the two monads $\mathcal{A}(I_a) \subset \mathcal{A}(I_b)$ uniquely determines a "canonical" split. The formula for this type factor \mathcal{N}_c which is functorially determined by the two monads reads

$$\mathcal{N}_c = \mathcal{A}(I_a) \vee J\mathcal{A}(I_a)J = \mathcal{A}(I_b) \wedge J\mathcal{A}(I_b)J \tag{23}$$

i.e. it is the operator algebra generated by the monad $\mathcal{A}(I_a)$ and its image under an antiunitary involution J which comes from the modular theory of the standard pair $(\mathcal{A}(I_a) \vee \mathcal{A}(I'_b), \Omega_{vac})$. In case the inclusion is split one can show that the algebra \mathcal{N}_c defined by this formula is really a type I factor in terms of which H and $B(H)$ tensor-factorizes. The advantage of this canonical choice is that it maintains the covariance under spacetime transformations, in this case the conformal covariance. Since there are many more intermediate type I subfactors with "fuzzy boundaries" than classical geometric boundary conditions any comparison with classical theory has its limitation; but if one looks for an analogy for the canonical functorial determination one may think perhaps of free boundary condition.

We are interested in the density matrix ρ which is obtained by the restriction of the vacuum state to \mathcal{N}_c , a concept which was not available on $\mathcal{A}(I_a)$ since monads have no density matrix states (and a fortiori no pure states). Note that ρ represents a thermal Gibbs state; the thermal KMS aspect is a property of any algebra which is either (sharply) localized or contained in a localized algebra as $\mathcal{N}_c \subset \mathcal{A}(I_b)$ and KMS states on type I algebras are Gibbs states. The

Hamiltonian is a operator in the factor space and can be read off from ρ i.e. it is an operator whose localization is inside I_b .

It is this step which replaces the somewhat artistic arguments based on functional integrals, the rest we take from the innovative and inspiring work of condensed matter physicists who use the field theoretic setting of factorizing models. In spite of the intrinsicness in the definition of ρ , I would presently not be able to write down an explicit formula for the canonical $\rho(b_1, a_1, a_2, b_2) \in \mathcal{A}(I_b)$ even though it is conformally covariant according to its functorial construction. But if one wants to extract the entropy from that thermal density matrix one may first use the *replica trick* to compute $\text{tr} \rho^n$ for $n=1,2,\dots$ and from there a representation of the entropy [30] in terms of an differentiation with respect to n at $n=0$. After legitimizing the uniqueness of the analytic continuation in n by checking the prerequisites of Carlson's theorem one obtains

$$s = -\text{tr} \rho \ln \rho = \frac{d}{dn} \text{tr} \rho^n |_{n=0} \quad (24)$$

The conformal invariance of these traces follows from the conformal covariance of ρ which in turn is a result of the functoriality of its construction in terms of conformally covariant algebras and the conformal invariance of the vacuum. As in the previous section this forces the traces and hence the entropy to be a function of the cross ratio of the four end-points.

In order to avoid confusions it should be stressed that these four points are *not* to be thought of as end points of localization regions but rather as parameters which designate a sharp localization region I_a together with an attenuation region for vacuum polarization given by the complement $I_b \setminus I_a$. The shape of the fuzzy attenuation cloud is completely fixed by the canonicity of the above split procedure in terms of the modular object associated with the canonical split of the inclusion of two monads $(\mathcal{A}(I_a) \subset \mathcal{A}(I_b), \Omega)$.

This is of course much more than the method of the previous section can deliver because the thermodynamic limit approximation by (relativistic) boxes can only be trusted in the leading volume (here length) proportionality which according to the previous section passes in chiral theories to the logarithm of the in the attenuation length $\varepsilon = \frac{1}{r}$ (with r given by the cross ratio below (30)²¹) via a conformal transformation to the logarithm). The higher corrections from vacuum polarizations are only accounted for by the split property and the associated canonical attenuation picture.

Such a simple correspondence between quantum heat bath- and quantum localization- thermality is only valid in chiral theories. Whereas this is not sufficient to relate heat bath and localization aspects in higher dimensional QFTs, it does just that for the holographic projections.

Unfortunately the present state of mathematical technology in operator algebras only permits to compute the leading term in the vanishing attenuation length i. e. in praxis one presently does not obtain more than in the previous

²¹In fact the conformal invariance of the chiral entropy permits to generalize the thermodynamic limit by limits in which the right and left hand side approach infinity with different velocities.

section. But since the method is quite interesting and allows us to make contact with recent results from condensed matter physics as in ([30] and references cited therein), we will present it in the sequel.

The next step in the derivation consists in the use of the *replica trick*. In the algebraic setting one starts from an n -fold tensor product of a chiral observable algebra on the circle. The following two formulas denote cyclic and permutation orbifold associated to the tensor product.

$$\begin{aligned} (\mathcal{A} \otimes \mathcal{A} \otimes \dots \otimes \mathcal{A})^{\mathbb{Z}_n} \\ (\mathcal{A} \otimes \mathcal{A} \otimes \dots \otimes \mathcal{A})^{P_n} \end{aligned} \quad (25)$$

whose construction requires the split property. It was introduced in [32] as an auxiliary tool to analyze problems with multi-interval inclusions. The second line denotes the closely related permutation orbifold whose irreducible representations are similar. The representation theory for tensor products is defined with the above split map but in order to come to a splitting situation we first apply a map which transforms an interval $I \subset S$

As usual the Riemann surface associated with $\sqrt[n]{z}$ is the n -fold ramified cover of $\mathbb{C} \setminus \{0\}$. We may use this as for the definition in order to map its n -fold ramified covering of the Moebius group into the following subgroup of $\text{Diff}(S^1)$ formally written as

$$z \rightarrow \sqrt[n]{\frac{\alpha z^n + \beta}{\beta z^n + \bar{\alpha}}} \quad (26)$$

The representations of the $Z(n)$ orbifold are constructed from the n right inverses of $f(z) = z^n$ which are injective maps g_0, g_1, \dots, g_{n-1} of $\mathbb{R} \rightarrow S^1$ which remain continuous at $\pm\infty$. On each interval $I \subset \mathbb{R}$ these maps are unitarily implemented and the resulting net $\Phi_{g_i, I}(\mathcal{A})$ can be used to define a representation of the tensor product algebra $\mathcal{A}(I) \otimes \dots \otimes \mathcal{A}(I)$ as

$$\pi_{f, I} \equiv \chi_I \cdot (\Phi_{g_0, I} \otimes \dots \otimes \Phi_{g_{n-1}, I}) \quad (27)$$

where χ_I is the natural isomorphism from $\mathcal{A}(I) \otimes \dots \otimes \mathcal{A}(I)$ to $\mathcal{A}(I_0) \vee \dots \vee \mathcal{A}(I_{n-1})$ from the canonical implementation of the split property. The net $\pi_{f, I}$ defines a soliton of $\mathcal{A}_0 \otimes \dots \otimes \mathcal{A}_0$ where the subscript is a reminder that the circle has been punctured at ∞ .

It turns out that the restriction to the cyclic orbifold i.e. the restriction

$$\tau_f \equiv \pi_f|_{(\mathcal{A} \otimes \dots \otimes \mathcal{A})^{\mathbb{Z}_n}} \quad (28)$$

has an extension to the full circle i.e. is a conformal field theory (indicated by omitting the subscript). It is quite common that a soliton representation passes to an ordinary representation. In the case at hand the irreducible soliton representation decomposes into a direct sum of n diffeomorphism covariant representations $\tau_f^{(0)}, \dots, \tau_f^{(n-1)}$ whose statistical dimension and scale dimensions

(of their generating fields) were determined in [32]. The anomalous spin spectrum can be read off directly from the embedding of the n -fold covering of the Moebius group into the $\text{Diff}(S^1)$ (26). For the following we only need the lowest scale dimension is

$$d_n = \frac{n^2 - 1}{12n} c \quad (29)$$

where c is the Virasoro constant.

The purpose of the orbifold representation in the present context is to identify the n^{th} power of the I_b -localized density matrix ρ^n with an operator in the τ_f representation and to extract information of the singular behavior for coalescent points when $I_b \rightarrow I_a$ by using the fact that the singularities of the branch points of this singular limit are determined by the lowest dimensional "twist" fields of the $Z(n)$ orbifold with dimension (29). This is precisely what Cardy et al. [30] arrive after (metaphoric) use of functional arguments in order to implement the replica trick.

The remaining steps are identical to theirs. For the cross ratio r we choose

$$r = \frac{(a_2 - a_1)(b_2 - b_1)}{(a_1 - b_1)(b_2 - a_2)} \quad (30)$$

which becomes singular in the limit $I_b \rightarrow I_a$. Unfortunately $\text{tr} \rho^n$ is a function of r which, although uniquely fixed in terms of $I_a \subset I_b$, is presently out of reach of our computational abilities. Its singular behavior leads to the formula

$$\begin{aligned} \text{tr} \rho^n &= r^{\frac{n^2-1}{24n}c} F_n(r) \\ F_1(r) &= 1 \end{aligned} \quad (31)$$

where the singular branch point behavior has been split off. Assuming finiteness of the derivative $\frac{d}{dn} F_n(r)|_{n=1} \equiv G(r)$ at $r \rightarrow \infty$ one obtains the limiting formula of the condensed matter literature

$$\begin{aligned} -\text{tr} \rho \ln \rho &= \frac{c}{12} \ln \frac{(a_2 - a_1)^2}{\varepsilon^2} \\ \varepsilon &= a_1 - b_1 = b_2 - a_2 \rightarrow 0 \end{aligned} \quad (32)$$

This is not the first time I have used the split property for the calculation of localization entropy. In [33] I used a formula for the unitary implementation of the splitting transformation which is limited to free fields. The resulting leading logarithmic dependence in the attenuation depth of the vacuum polarization led me to expect that this behavior is generic. In order to show this I looked for other ways and found the relation with the thermodynamic limit formula of the previous section. But it was only after I recently became aware of the work in condensed matter physics that I succeeded to complete my old program of computing at least the leading behavior of the canonically defined localization entropy.

In order to avoid misunderstanding, it is not our intention to compete with the beautiful results obtained about localization entropy in condensed matter physics [30]; my main point is a methodological. Functional integrals, even in cases where they exist and are backed up by measure theory, as for super-renormalizable QFT (finite wave function renormalization), are unsuitable for the description of localized subtheories as needed to define localization entropy or localization energy. They are in fact blind against the thermal manifestations resulting from the local monad structure of localized algebras as compared to the quantum mechanical structure of the global algebra. Monads only occur in QFT and not in QM and functional integrals have the same appearance in QFT and QM.

In [30] the functional integral representation is only used in a metaphoric way in order to implement the replica idea. All the calculations are done in the bootstrap-formfactor setting. Indeed the setting of functional integrals is the most marvelous metaphoric instrument of QFT. For the purpose for which it is used by Cardy et al. it is particularly suitable, and the fact that factorizing models are outside the range of validity of functional integral representations will not leave a pragmatically inclined quantum field theorist sleepless as long as his consistency checks work.

But even the staunchest pragmatist cannot fail to perceive the deep irony which lies in the fact that in those cases where the functional integral is exact, namely in QM, it is not possible to teach a normal course on QM using only functional integration²²; on the other hand modern textbooks tend to equate the definition of QFT with functional integral quantization despite its metaphoric content. As a result there are particle physicists who think that perturbative divergencies and their renormalization via cutoffs or regulators are intrinsic attributes of QFT. It is often not noticed that the causal approach has shown already many decades ago that the principles of QFT implemented iteratively, starting with the Wick-ordered lowest order interaction density, lead to a finite formulation which however in certain cases has an increasing number of free parameters (nonrenormalizability) and as a result ceases to be useful.

Though in most cases (including the present one) one does not really have to rely on metaphors, their use often significantly facilitate the communication between particle physicists. Writing a specific functional integral on a blackboard generates a strain of associations which is generally sufficient to initiate a meaningful discussion; it is hard to think of any other compact effective way. The metaphoric power is strongest when the setting is used as a vehicle to discover new mathematical structures as it was first done in the work by Atiyah and Witten in the 70/80s.

By during the last two decades the limitations of this metaphoric power hav-

²²It is an interesting intellectual exercise and a test of one's conceptual understanding of QM to contemplate how quantum theory would have evolved if Feynman's approach would have appeared before Heisenberg's. The idea is not as harbrained as it appears at first sight because the functional integral approach is conceptually much closer to the old Bohr-Sommerfeld QM than Heisenberg's rather abstract setting. For calculating quasiclassical approximations the Feynman approach is the most elegant and effective starting point.

ing become increasingly evident. The local covariance principle in the context of QFT in generic curved spacetimes is not even metaphorically compatible with a functional integral setting, and neither are QFT with braid group statistics as chiral models. Also there are factorizing models which are metaphorically consistent with a functional representation most of them are not; and even if they are, as e.g. the Sine-Gordon model, the functional metaphor is of no help in its solution.

The present state of QFT is that of an ongoing paradigmatic change where at the end one expects to arrive at a setting which parallels the conceptual cohesion and the mathematical precision of the operator formalism of QM. During this transition time the functional integral setting will continue to be the source of new ideas. There is no harm in using its suggestive power as long as one remains aware that it is of a metaphoric nature.

There is an interesting conceptual difference which remains between my work on localization entropy and the work by condensed matter physicists even though both used a QFT framework. From my point of view the use of the terminology "cutoff" in connection with localization entropy is not helpful because it creates the wrong association; for this reason I have avoided it ever since I started my work at the beginning of this decade [33] and used instead the concept of an attenuation length ε . Hardly anybody would associate the divergent volume factor which appears in the thermodynamic limit of thermal systems with a cutoff, yet the attenuation length parameter of the vacuum polarization cloud is nothing but a conformal transform of the length factor L which appears in the thermodynamic limit of a chiral heat bath QFT.

Cutoffs in QFT are uncontrollable changes of theories caused by cutting out the high energy contributions in certain integrations in the hope that despite the uncontrollable change certain numerical quantities of interests may change only little. The notion of attenuation length for localization-caused vacuum polarization on the other hand is a rigorous concept *within* each fixed QFT model.

7 The conceptual-philosophical basis of a modular-based approach, messages for QG

A radically different approach to QFT as the present one, which substitutes any kind of quantization parallelism to classical fields by completely autonomous concepts should come with different conceptual-philosophical message of what constitutes the essence of QFT. Indeed the scenario of holography and its inversion via reconstruction wedge asks for a different philosophical setting than that of Lagrangian quantization. Whereas similar to QM the latter harmonizes with a Newtonian view of quantum matter as something that fills spacetime, the monad structure of local operator algebras in QFT and their intersection and generating properties require Leibniz's more abstract view of spacetime as an ordering device such that holography is a radical change of this ordering device.

The underline the radical aspect of this new viewpoint we refer to [15] where it was pointed out that quantum matter *together* with its spacetime symmetries as well as all its inner symmetries can be encoded into the *position of a finite number of monads* (i.e. copies of the unique abstract monad) *in a common Hilbert space* [15]. Even the kind of quantum matter (hadrons, leptons, photons) is resolved in terms of positioning, with other words the ultimate reality of QFT is relative positioning of a finite number of monads in a Hilbert space. What makes this different perspective of QFT so interesting is that it is completely rigorous as well as conservative. It does not replace or add any physical principle, yet it implies a strong change of paradigm. This is the strongest indication yet that QFT is still a very young theory with expected changes and certainly nowhere near to its closure.

In the form as applied in this paper modular theory was used to address the localization problems and symmetries of QFT in Minkowski spacetime. It is natural to ask whether these ideas using modular groups can be applied in the more general context of QFT in CST. A more modest question in this direction would be to understand whether the $\text{Diff}(S^1)$ symmetries beyond Moebius symmetries which do not preserve the vacuum can be obtained by modular methods (i.e. without assuming the existence of an energy-momentum tensor which for chiral theories originating from holography is in any case not a reasonable assumption). Preliminary studies indicate that this is the case if one relaxes some of the modular concepts.

An important issue is how to view the generic area proportionality of localization entropy of quantum matter on null-horizons in connection with Bekenstein's classical area behavior in Einstein-Hilbert like classical field theories. The standard argument consists in using Bekenstein's quantum re-interpretation as a key to learn something about the elusive QG. Whatever one wants to use it for, one can certainly not claim that the entropy area law is direct evidence of manifestation of QG. The thermal aspects of Hawking radiation as well as the area proportionality of entropy are perfectly describable in the setting of QFT in CST; no appeal to a still elusive QG is necessary.

The formation of a black hole through a collapsing star, as envisaged by Hawking [18] and described in more detail within an algebraic QFT setting by Haag and Fredenhagen [34], is outside the static equilibrium thermodynamic setting. For such stationary nonequilibrium states the recent notion of entropy flux in the operator algebra setting [35] may be more appropriate.

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References

- [1] R. Haag, J. A. Swieca, Commun. Math. Phys. **1**, (1965) 308
- [2] B. Schroer, *Particle physics in the 60s and 70s and the legacy of contributions by J. A. Swieca*, arXiv:0712.0371
- [3] D. Buchholz and E. H. Wichmann, Commun. math. Phys. **106**, (1986) 321

- [4] D. Buchholz and P. Junglas, *commun. Math. Phys.* **121**, (1989) 255
- [5] S. Åks; *Journ. Math. Phys.* **6** (1965) 516
- [6] M. Karowski, H. J. Thun, T. T. Truong, and P. Weisz, *Phys. Lett. B* **67**, (1977) 321
- [7] B. Schroer, *Ann. Phys.* **295**, (1999) 190
- [8] H. J. Borchers, D. Buchholz and B. Schroer, *Commun. Math. Phys.* **219**, (2001) 125, hep-th/0003243
- [9] G. Lechner, *An Existence Proof for Interacting Quantum Field Theories with a Factorizing S-Matrix*, math-ph/0601022
- [10] H. Leutwyler, *Acta Phys. Austr.*, Suppl.V. H. Leutwyler in: *Magic without Magic* (J. R. Klauder ed.). Freeman, 1972. F. Jegerlehner, *Helv. Phys. Acta* **46**, (1974) 824
- [11] W. Driessler, *Acta Phys. Austr.* **46**, (1977) 63
- [12] B. Schroer, *Class.Quant.Grav.* **23** (2006) 5227, hep-th/0507038 and previous work cited therein
- [13] D. Guido. R. Longo and H. W. Wiesbrock, *Commun. math. Phys.* **192**, (1998) 217
- [14] J. Yngvason, *Rept.Math.Phys.* **55** (2005) 135, math-ph/0411058
- [15] B. Schroer, *Localization and the interface between quantum mechanics, quantum field theory and quantum gravity*, arXiv:0711.4600
- [16] S. J. Summers, *Tomita-Takesaki Modular Theory*, math-ph/0511034
- [17] J. Mund, B. Schroer and J. Yngvason, *Commun. Math. Phys.* **268**, (2006) 621
- [18] S. W. Hawking, *Commun. Math. Phys.* **43**, (1975) 199
- [19] G. L. Sewell, *Ann. Phys.* **141**, (1982) 201
- [20] M. Joerss, *Lett. Math. Phys.* **38**, (1996) 257
- [21] B. Schroer, T.T. Truong and P. Weisz, *Phys. Lett. B* **63**, (1976), 422
- [22] M. Karowski and P. Weisz, *Nucl. Phys. B* **139**, (1978) 455
- [23] H. Babujian and M. Karowski, *Phys.Lett. B* **575** (2003) 144, arXiv:hep-th/0309018
- [24] B. Schroer, *Ann. Phys.* **307**, (2003) 421
- [25] F. Coester and W. N. Polyzou, *Phys. Rev. D* **26**, (1982) 1348 and references therein

- [26] F. Smirnov, Adv. Series in Math. Phys. 14, World Scientific, 1992
- [27] S. J. Summers and R. Verch, Lett. Math. Phys. **37**, (1996) 145
- [28] B. Schroer, Class.Quant.Grav. **24** (2007), 1
- [29] J.J Bisognano and E. H. Wichmann, Journal of mathematical physics **17**, (1976) 303
- [30] J.L. Cardy, O.A. Castro-Alvaredo and B. Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, hep-th/0706.3384 and the previous literature cited therein
- [31] N. M. Nikolov, I. T. Todorov, *Lectures on Elliptic Functions and Modular Forms in Conformal Field Theory*, arXiv:math-ph/0412039
- [32] R. Longo and F. Xu, Commun. Math. Phys. **251**, (2004) 321
- [33] B. Schroer, IJMPA **18**, (2002) 1671 and references therein
- [34] K. Fredenhagen and R. Haag, Commun. Math. Phys. **127** (1990) 273
- [35] V. Jakšić V and C-A. Pillet, Commun. Math. Phys. **217** (2001) 285